Malaysian Journal of Mathematical Sciences 8(S): 153- 161 (2014)
Special Issue: International Conference on Mathematical Sciences and Statistics 2013 (ICMSS2013)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

# Spectral Homotopy Analysis Method for Solving Nonlinear Volterra Integro Differential Equations

**[2*]Zohreh Pashazadeh Atabakan, [2]Aliasghar Kazemi Nasab and [1,2]Adem Kilicman**

*[1]Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*

*[2]Department of Mathematics, Faculty of Science, Universiti Putra Malaysia,43400 UPM Serdang, Selangor,Malaysia*

*E-mail: pashazadeh2002@yahoo.com*

**Corresponding author*

## ABSTRACT

In this paper we proposed Spectral Homotopy analysis method to solve nonlinear Volterra integro-differential equations .Some examples are given to approve the efficiency and the accuracy of the proposed method. The SHAM results show that the proposed approach is quite reasonable when compared to homotopy analysis method and exact analytical solutions.

Keywords: Homotopy analysis method, spectral homotopy analysis method, Clenshaw-Curtis quadrature formula, Volterra integro-differential equations.

## 1. INTRODUCTION

In recent years a growing interest can be seen in integro-differential equations (Karamete *et al.* (2002), Darania *et al.* (2007), Parand *et al.*(2011) and Pashazadeh *et al.* (2013)). integro-differential equations have described many problems of applied sciences. This kind of equations are usually difficult to solve analytically so mathematicians are interested to introduce a numerical method to obtain an efficient approximate solution. In (Liao (1997)) Homotopy analysis method was introduced by Liao. This method is based on topology and is power full for solving problem with strong nonlinearity.This method is more accurate and powerful than other perturbation methods. In this method we can choose different base function to approximate a nonlinear problem. Spectral homotopy analysis method is a modification of homotopy analysis method. Motsa *et al*. (2010) combines the HAM with a numerical technique to solve higher order deformation equation and recalled Spectral homotopy analysis method. Some nonlinear ordinary differential equations were solved using this new approach by Motsa *et al*. (2010). Pashazadeh *et al.* (2012, 2013) applied this method for solving linear Volterra and nonlinear Fredholm integro-differential equations. In this paper, we apply spectral homotopy analysis method (SHAM) to solve nonlinear Volterra type of integro-differential equations. Volterra integro-differential equation is given by

$$\begin{cases} \sum_{j=0}^{2} a_j(x) v^{(j)}(x) = g(x) + \mu \int_{-1}^{x} k(x,t)[v(t)]^r \, dt, \\ v(-1) = v(1) = 0, \end{cases} \tag{1}$$

The paper is organized in the following way: Spectral homotopy analysis method for solving nonlinear Volterra integro-differential equations is presented in next Section . Then some numerical examples are presented to show the accuracy and efficiency of this method. Finally, concluding remarks are given.

## 2. SPECTRAL-HOMOTOPY ANALYSIS SOLUTION

Consider the non linear Volterra integro-differential equation (1). A linear operator is defined as following

$$L(\zeta(x;q)) = \sum_{j=0}^{2} a_j(x) \frac{\partial^j \zeta(x;q)}{\partial x^j} \tag{2}$$

where $\zeta(x;q)$ is an unknown function and $q \in [0,1]$ is the embedding parameter. The zeroth-order deformation equation is given by

$$(1-q)L[\zeta(\gamma;q) - v_0(\gamma)] = qh(N[\zeta(\gamma;q)] - g(\gamma)), \tag{3}$$

$h$ is the non-zero convergence controlling auxiliary parameter and $N$ is a nonlinear operator given by

$$N[\zeta(\gamma;q)] = \sum_{j=0}^{2} a_j(\gamma) \frac{\partial^j \zeta(\gamma;q)}{\partial \gamma^j} - g(\gamma) - \mu \int_{-1}^{\gamma} k(\gamma,t) \zeta^r(t) dt. \tag{4}$$

Differentiating Eq.(3) $m$ times with respect to the embedding parameter $q$ and then setting $q=0$ and finally dividing them by $m!$, we have the so called $m$ th-order deformation equation

$$L[v_m(\gamma)-\chi_m v_{m-1}(\gamma)] = hR_m, \tag{5}$$

subject to boundary conditions

$$v_m(-1) = v_m(1) = 0, \tag{6}$$

where

$$R_m(\gamma) = \sum_{j=0}^{2} a_j(\gamma)\frac{\partial^j \zeta(\gamma;q)}{\partial \gamma^j} - g(\gamma)(1-\chi_m) - \mu\int_{-1}^{\gamma} k(\gamma,t)\zeta^r(t)dt. \tag{7}$$

The initial approximation that is used in the higher-order equation (5) is obtained on solving following Equation

$$\sum_{j=0}^{2} a_j(x)v_0^{(j)}(x) = g(x) \tag{8}$$

subject to boundary conditions

$$v_0(-1) = v_0(1) = 0, \tag{9}$$

Eqs (8)-(9) are solved using the Chebyshev pseudo-spectral method. In this method $v_0(\gamma)$ is approximated by a truncated series of Chebyshev polynomial of the following form

$$v_0(\gamma) \approx v_0^N(\gamma_j) = \sum_{k=0}^{N} \hat{v}_k T_k(\gamma_j), \quad j = 0,...,N, \tag{10}$$

where $T_k$ is the $k$ th Chebyshev polynomials, $\hat{v}_k$ are coefficients and Gauss-Lobatto collocation points $\gamma_0, \gamma_1, \ldots, \gamma_N$, which are the extreme of the $N$ th order Chebyshev polynomial, defined by

$$\gamma_j = \cos\left(\frac{\pi j}{N}\right). \tag{11}$$

Derivatives of the functions $v_0(\gamma)$ at the collocation points are represented as

$$\frac{d^s v_0(\gamma_k)}{d\gamma^s} = \sum_{j=0}^{N} D_{kj}^s v_0(\gamma_j), \quad k = 0,...,N \tag{12}$$

where $s$ is the order of differentiation and $D$ is the Chebyshev spectral differentiation matrix (Don *et al*. (1995)). Substituting Eqs (10)-( 12) into (8) will result in

$$\mathbf{B}\mathbf{V}_0 = \mathbf{G} \tag{13}$$

subject to the boundary conditions

$$v_0(\gamma_0) = v_0(\gamma_N) = 0, \tag{14}$$

where

$$\begin{aligned} \mathbf{B} &= \sum_{j=0}^{2} \mathbf{a}_j \mathbf{D}^j, \\ \mathbf{V}_0 &= [v_0(\gamma_0), v_0(\gamma_1), ..., v_0(\gamma_N)]^T, \\ \mathbf{G} &= [g(\gamma_0), g(\gamma_1), ...g(\gamma_N)]^T, \\ \mathbf{a}_r &= diag(a_r(\gamma_0), a_r(\gamma_1), ...a_r(\gamma_N)). \end{aligned} \tag{15}$$

The values of $v_0(\gamma_i),\ i = 0,...,N$ are determined from the equation

$$\mathbf{V}_0 = \mathbf{B}^{-1}\mathbf{G}, \tag{16}$$

which is the initial approximation for the SHAM solution of the governing equation (1). The Chebyshev pseudo-spectral transformation is appliedon equations (5)-(7) and obtain the following result

$$\mathbf{B}\mathbf{V}_m = (\chi_m + h)\mathbf{B}\mathbf{V}_{m-1} - h[\mathbf{S}_{m-1} + (1-\chi_m)\mathbf{G}], \tag{17}$$

subject to the boundary conditions

$$v_m(\gamma_0) = v_m(\gamma_N) = 0, \tag{18}$$

where $\mathbf{B}$ and $\mathbf{G}$ were defined in and

$$\mathbf{V}_m = [v_m(\gamma_0), v_m(\gamma_1), ..., v_m(\gamma_N)]^T,\ \ \mathbf{S}_m = \int_{-1}^{\gamma} k(\gamma,t)[\mathbf{V}_m]^r\, dt. \tag{19}$$

To implement the boundary condition (17) we delete the first and the last rows of $\mathbf{S}_{m-1}$, $\mathbf{G}$ and the first and the last rows and columns of $\mathbf{B}$. Finally this recursive formula can be written as follows

$$\mathbf{V}_m = (\chi_m + h)\mathbf{V}_{m-1} - h\mathbf{B}^{-1}[\mathbf{S}_{m-1} + (1-\chi_m)\mathbf{G}_{m-1}], \tag{20}$$

with starting from the initial approximation we can obtain higher order approximation $\mathbf{V}_m$ for $m \geq 1$ recursively. To compute the integral in equation (19) we use the Clenshaw-Curtis quadrature formula: (Davis *et al.* (1970))

$$\mathbf{S}_m(\gamma) = \int_{-1}^{\gamma} k(\gamma, t, \tilde{\mathbf{V}}_m)dt = \sum_{j=0}^{N} w_j k(\gamma, \gamma_j, \tilde{\mathbf{V}}_m). \tag{21}$$

where the nodes $\gamma_j$ are given by equation (11) .

## 3. NUMERICAL EXAMPLES

In this section we apply the technique described in this paper Section 3 to some illustrative examples of nonlinear Volterra integro-differential equations.

**Example 1.** Consider the second order the nonlinear Volterra integro-differential equation:

$$v''(x) + v'(x) + 3v(x) = f(x) + \int_{-1}^{x} xtv'(t)v(t)dt, \tag{22}$$

$$v(-1) = v(1) = 0,$$

where

$$f(x) = -\frac{2}{5}x^6 + \frac{2}{3}x^4 + 3x^2$$

We employ SHAM to solve this example. In Figure 1, there is a comparison between exact solution and the 15th order approximate solution. In Table 1 is illustrated a comparison of absolute errors against the SHAM solutions.

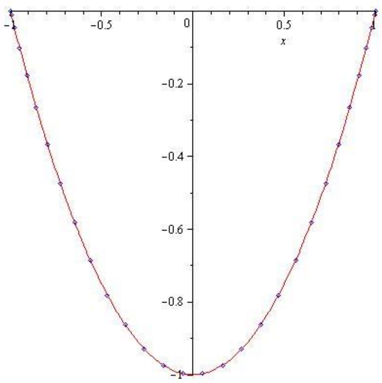


Figure1: Comparison of Exact solution and $y_{15}$

TABLE 1: Absolute errors of Ex.1 against the SHAM solutions with h=-1.8

| x | SHAM 15th order | SHAM 23th order | Numerical |
|---|---|---|---|
| 0.1 | $7.9\times10^{-5}$ | $2.7\times10^{-7}$ | 0.99 |
| 0.2 | $6.6\times10^{-5}$ | $4.2\times10^{-7}$ | 0.96 |
| 0.3 | $5.7\times10^{-5}$ | $5.0\times10^{-7}$ | 0.91 |
| 0.4 | $4.6\times10^{-5}$ | $4.5\times10^{-7}$ | 0.84 |
| 0.5 | $3.7\times10^{-5}$ | $1.7\times10^{-7}$ | 0.75 |
| 0.6 | $3.0\times10^{-5}$ | $4.0\times10^{-7}$ | 0.64 |
| 0.7 | $2.6\times10^{-5}$ | $1.2\times10^{-6}$ | 0.51 |
| 0.8 | $2.2\times10^{-5}$ | $1.8\times10^{-6}$ | 0.36 |
| 0.9 | $1.5\times10^{-5}$ | $1.8\times10^{-6}$ | 0.19 |
| 1.0 | 0.0 | 0.0 | 0.0 |

**Example 2.** Consider the second order nonlinear Volterra integro-differential equation:

$$v''(x)+4v'(x)=f(x)+\int_{-1}^{x} xtv^2(t)dt, \tag{23}$$
$$v(-1)=v(1)=0,$$

where

$$f(x)=-\frac{1}{6}x^7+\frac{1}{2}x^5-\frac{1}{2}x^3-8x^2+\frac{1}{6}x-2$$

We employ HAM and SHAM to solve this example. As it is illustrated in Figure 2 and Figure 3, the rate of convergence in SHAM is faster than HAM. In Figure 4, it is found that when $-2\le h\le 0$ and $-1\le h\le 0$, the SHAM solution and HAM solution converge to the exact solution, respectively. There is a comparison against the HAM and the SHAM solutions in Table 2.

TABLE 2: Numerical result of Example 2 against the HAM and the SHAM solutions with h=−1

| x | SHAM 2nd order | SHAM 10th order | HAM 12th order | Numerical |
|---|---|---|---|---|
| 0.1 | 0.99000082 | 0.99000000 | 0.99000046 | 0.99 |
| 0.2 | 0.96000066 | 0.96000000 | 0.96000060 | 0.96 |
| 0.3 | 0.91000051 | 0.91000000 | 0.91000063 | 0.91 |
| 0.4 | 0.84000039 | 0.84000000 | 0.84000051 | 0.84 |
| 0.5 | 0.75000034 | 0.75000000 | 0.75000042 | 0.75 |
| 0.6 | 0.64000038 | 0.64000000 | 0.64000055 | 0.64 |
| 0.7 | 0.51000055 | 0.51000000 | 0.51000080 | 0.51 |
| 0.8 | 0.36000087 | 0.36000000 | 0.36000084 | 0.36 |
| 0.9 | 0.19000036 | 0.19000000 | 0.19000045 | 0.19 |
| 1.0 | 0.0 | 0.0 | 0.0 | 0.0 |

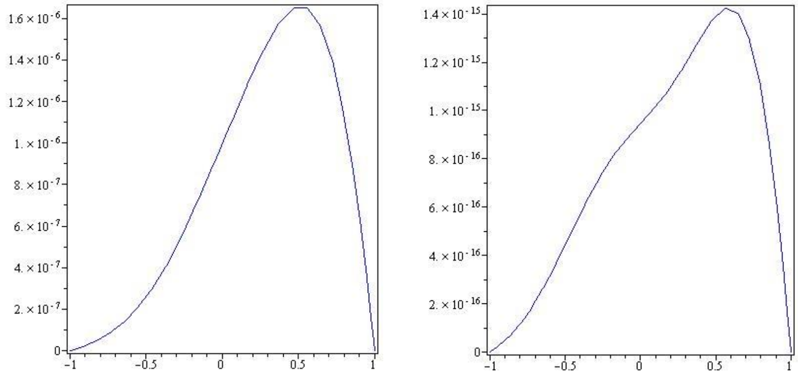


Figure 2: The absolute errors of second order (Left) and seventh order, (Right) SHAM.

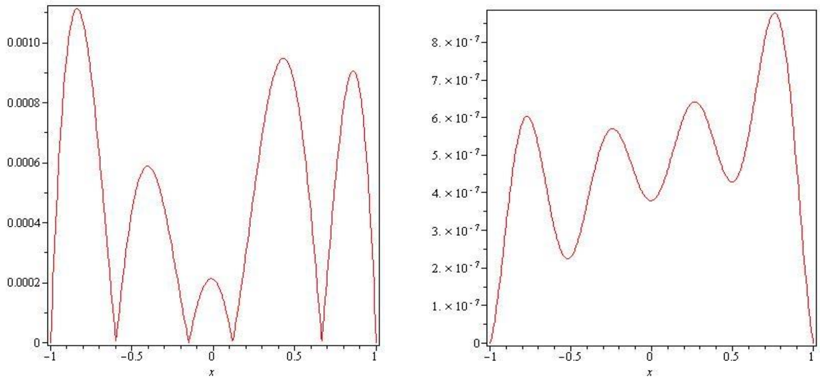


Figure 3: The absolute errors of fifth order (Left) and seventh order, (Right) HAM.

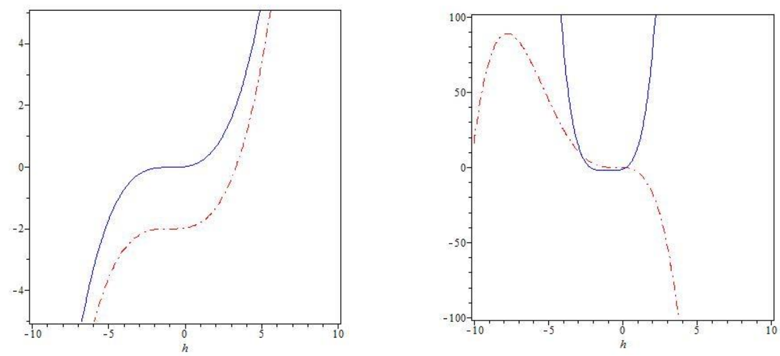


Figure 4: The graph of h-curve for 10th order (Left) SHAM, (Right) HAM.

## 4. CONCLUSIONS

In this paper we presented the application of spectral homotopy analysis method(SHAM) for solving nonlinear Volterra integro differential equations. A comparison was made between exact analytical solutions and numerical results from the spectral homotopy analysis method and homotopy analysis method (HAM) solutions. The numerical results indicate that in SHAM the rate of convergence is faster than HAM. For example, in Example 2 we found that the seventh order SHAM approximation sufficiently give a match with the numerical results up to sixteenth decimal places. In contrast, HAM solutions have a good agreement with the numerical results in this order only up to seventh decimal places. As it is shown in the Figure 4 the range of admissible h values is much wider in SHAM than in HAM.

## REFERENCES

Darania, P. and Ebadian, A. (2007). A method for the numerical solution of the integro-differential equations. *Appl.Math. Comput*. **188** : 657-668.

Davis, P. J. and Rabinowits, P. (1970). *Method of Numerical Integration*. 2nd ed. London /New York: Academic Press.

Don, W. and Solomonoff , A. (1995) .Accuracy and speed in computing the cheby sheve collocation derivative. *SIAM Sci Computer*. **16**(6):1253-1268.

Karamete, A. and Sezer, M. (2002). A Taylor collocation method for the solution of linear integro-differential equations. *Intern. J. Computer Math.* **79**(9): 987-1000.

Liao, S. J. (1992). The proposed homotopy analysis technique for the solution of nonlinear problems. Ph.D. Thesis, Shanghai Jiao Tong University.

Motsa, S. S., Sibanda, P. and Shateyi, S. (2010). A new spectral-homotopy analysis method for solving a nonlinear second order BVP. *Commun. Nonlinear Sci. Numer Simulat*. **15**: 2293-2302.

Motsa, S. S. Sibanda, P. and Shateyi, S. (2010). A new spectral-homotopy analysis method for the MHD Jeffery-Hamel problem. *Computer Fluids*. **39**: 1219-1225.

Motsa, S. S. Sibanda, P. (2010). A new algorithm for solving singular IVPs of Lane-Emden type, *in Proceedings of the 4th International Conference on Applied Mathematics, Simulation, Modelling, WSEAS International Conferences*, Corfu Island, Greece, 176-180.

Parand, A., Abbasbandy, S., Kazem, S. and Rad, J. A. (2011). A novel application of radial basis functions for solving a model of first-order integro-ordinary differential equation. *Commun Nonlinear Sci. Numer. Simulat.* **16**: 4250-4258.

Pashazadeh Atabakan, Z., Kazemi Nasab, A. and Kilicman, A. (2013). On solution of Fredholm integro-differential equations using composite Chebyshev finite difference method. *Abstract and applied analysis method*, doi.org/10.1155/2013/67436.

Pashazadeh Atabakan, Z., Kilicman, A. and Kazemi Nasab, A. (2012). On spectral homotopy analysis method for solving Volterra and Fredholm type of integro-differential equations. *Abstract and Applied Analysis*, doi:10.1155/2012/960289.

Pashazadeh Atabakan, Z., Kazemi Nasab, A., Kilicman, A and Zainidin K. Eshkuvatova. (2013). Numerical solution of nonlinear Fredholm integro-differential equations using Spectral Homotopy Analysis method. *Mathematial problems in engineering*, doi.org/ 10.1155/2013/674364.